The Boundary Value Problem for a Static 2D Klein–Gordon Equation in the Infinite Strip and in the Half-Plane[*]

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Abstract

We provide explicit formulas for the Green function of an elliptic PDE in the infinite strip and the half-plane. They are expressed in elementary and special functions. Proofs of uniqueness and existence are also given.

1 Motivation

This research is motivated by the multi-asset American option pricing problem(see e.g. [2]). Using the standard arbitrage theory framework it can be shown that the option prices are bounded by solutions of elliptic PDE problems. These problems have specific features: the domains are unbounded and the boundary functions are not smooth and do not vanish at infinity. We reduce the original PDE determining the price to a static Klein–Gordon equation (SKGE) in an infinite strip (for the alternative dual options) and in the half plane (the exchange and basket options). The boundary conditions of the corresponding problems are bounded Hölder functions. This leads to the following questions:

   i. Can we explicitly solve the boundary problems for the SKGE?

  ii. If this is the case, can we construct computer- friendly representations?

 iii. Is the obtained solution classical?

  iv. Is it unique?

To our knowledge, these aspects of the mentioned problems have not been studied in the literature. Even the answers to the two last questions seem not be obvious. The majority of the known results deal with bounded domains and one can not directly apply available theorems to the equations in the infinite strip and in the half-plane. Also, if the boundary conditions are not twice differentiable, then the smoothness of the solution is not clear. In this paper we give answers to all four questions. Our results are as follows:

   i. The boundary value problem in the infinite strip and in the half-plane allows a closed form solution.

  ii. Solutions can be represented in terms of elementary functions or in terms of special functions.

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iii. For all bounded boundary functions with Hölder property the solution is classical.

iv. The solution is unique.

We could find in the literature only a few related works. The paper [4] contains closed form formulas of boundary problem for the Laplace equation in the infinite strip. In the recent paper [6] there is a variety of Green functions for the homogeneous Poisson problems for SKGE in some unbounded domains, including the strip and the half-plane.

The paper is organized as follows. In Section 2 we obtain explicit formulas for the Green function in the case of infinite strip and formulate the main theorem on existence, uniqueness and closed form of the solution. In Section 3 we address the case of half-plane domain. All proofs are given in Section 4.

2 Problem in the infinite strip

2.1 Existence and uniqueness theorem

Let $\Pi^\pi = \mathbb{R} \times [0, \pi] = \left\{ (x, y) \in \mathbb{R}^2, \ y \in [0, \pi] \right\}$. Let $LV = \Delta V - r^2 V$ be an operator acting on the twice differentiable functions $V = V(x, y)$ defined in the interior of Π^π. Here Δ is the Laplacian and $r \in \mathbb{R}$. We consider the boundary value problem

$$\begin{cases} LV(x, y) = 0, & (x, y) \in \operatorname{int} \Pi^\pi, \\ V(x, 0) = \varphi(x), & x \in \mathbb{R}, \\ V(x, \pi) = 0, & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

The following theorem claims that under certain assumptions the problem (2.1) admits a unique classical solution and provides an explicit form for it.

Theorem 2.1. *Let H^λ be the space of Hölder functions of order $\lambda > 0$ and let φ be a bounded function from H^λ. Then the solution of the problem (2.1) exists in the classical sense, is unique, and allows the representation*

$$V(x, y) = \int_{\mathbb{R}} \varphi(u) G^\pi(x - u, y) du, \tag{2.2}$$

with the Green function

$$G^\pi(x, y) = \delta(x)\Theta(-y) + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{k \sin ky}{\sqrt{k^2 + r^2}} e^{-|x|\sqrt{k^2 + r^2}}. \tag{2.3}$$

where $\delta(x)$ is the Dirac delta-function and $\Theta(x) = I_{(0,\infty)}(x)$.

Corollary 2.1. *The Green function G_Δ^π for the Laplace equation $\Delta V = 0$ (i.e. for $r = 0$) has the representation (see [4]):*

$$G_\Delta^\pi(x, y) = \delta(x)\Theta(-y) + \frac{1}{2\pi} \frac{\sin y}{\cosh x - \cos y}. \tag{2.4}$$

Theorem 2.2. *The Green function G^π given by (2.3) has the integral representation*

$$G^\pi(x, y) = \delta(x)\Theta(-y) + \frac{|x|}{2\pi} \sin y \int_1^\infty \frac{J_0(|x|r\sqrt{t^2 - 1}) \sinh |x|t}{(\cosh xt - \cos y)^2} dt \tag{2.5}$$

where $J_0(z)$ is the Bessel function of zero order.

Corollary 2.2. *The Green function (2.3) can be represented in terms of the Green function G_Δ^π as follows:*

$$G^\pi(x,y) = G_\Delta^\pi(x,y) - r \int_{|x|}^\infty G_\Delta^\pi(t,y) \frac{J_1(r\sqrt{t^2 - x^2})t}{\sqrt{t^2 - x^2}} dt, \tag{2.6}$$

where $J_1(z)$ is the Bessel function of first order.

REMARK 2.1. The problem $LV = 0$ with boundary conditions $V(x,0) = 0$, $V(x,\pi) = \tilde\varphi(x)$ can be reduced to the problem (2.1 by the substitution $\tilde y = \pi - y$. Using the linearity of L we obtain the solution of the problem $LV = 0$ with the boundary conditions $V(x,0) = \varphi(x)$ and $V(x,\pi) = \tilde\varphi(x)$ as the sum of solutions of the problems in which one of the boundary conditions is a function equal to zero.

2.2 Green function: construction

In this subsection we derive the formulas (2.3) for the Green function. To this aim we introduce the interaction potential $P^\pi(x,y;u)$ defined as the limit

$$P^\pi(x,y;u) = \lim_{\varepsilon \to 0} P_\varepsilon^\pi(x,y;u),$$

The function $P_\varepsilon^\pi(x,y;u)$ is the solution in the distribution sense (see, e.g. [7]) of the boundary value problem

$$\begin{cases} LP_\varepsilon^\pi = 0, \\ P_\varepsilon^\pi|_{y=0} = e^{-\varepsilon(x-u)}\Theta(x - u), \\ P_\varepsilon^\pi|_{y=\pi} = 0. \end{cases} \tag{2.7}$$

We define the Green function as the partial derivative in x of P^π:

$$G(x - u, y) = \frac{\partial P^\pi(x,y;u)}{\partial x}. \tag{2.8}$$

Let us consider the Fourier transform in x of the potential P_ε^π:

$$v_\varepsilon(y;\xi,u) = \int_{\mathbb{R}} e^{i\xi x} P_\varepsilon^\pi(x,y;u) dx,$$

It solves, as a function of y, the two-point problem

$$\begin{cases} \dfrac{\mathrm{d}^2 v_\varepsilon}{\mathrm{d}y^2} - (\xi^2 + r^2)v_\varepsilon = 0, \\ v_\varepsilon(0;\xi,u) = \dfrac{e^{i\xi u}}{\varepsilon - i\xi}, \\ v_\varepsilon(\pi;\xi,u) = 0. \end{cases} \tag{2.9}$$

The solution has the form

$$v_\varepsilon(y;\xi,u) = \frac{e^{i\xi u} \sinh\left((\pi - y)\sqrt{r^2 + \xi^2}\right)}{(-i\xi + \varepsilon) \sinh\left(\pi\sqrt{r^2 + \xi^2}\right)}.$$

Making the inverse transform, we obtain the explicit formula for the potential P_ε^π:

$$P_\varepsilon^\pi(x,y;u) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\xi(u-x)} \sinh\left((\pi - y)\sqrt{r^2 + \xi^2}\right)}{(-i\xi + \varepsilon) \sinh\left(\pi\sqrt{r^2 + \xi^2}\right)} d\xi. \tag{2.10}$$

The integrand is an analytical function in the whole complex plane except zeros of functions

$$\xi + i\varepsilon = 0, \qquad \sinh\left(\pi\sqrt{\xi^2 + r^2}\right) = 0,$$
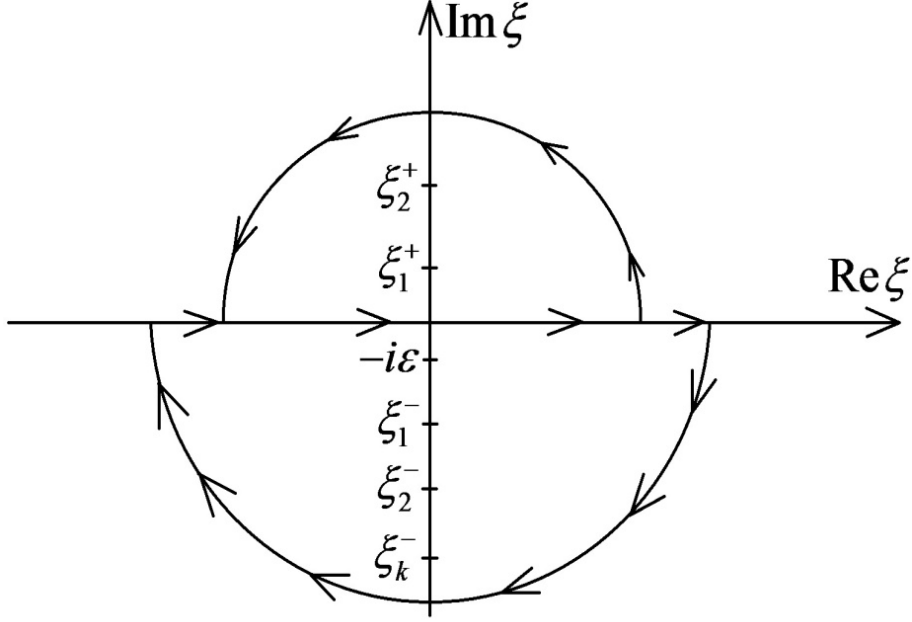
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Figure 1:   Integration loop for the strip case

that is except the points

$$\xi = -i\varepsilon, \quad \xi_k^{\pm} = \pm\sqrt{k^2 + r^2}.$$

Integrating along the loop depicted in Figure 1 and applying the Jordan lemma we obtain the infinite sum representation of (2.10)

$$
\begin{aligned}
P_\varepsilon^\pi(x,y;u) \quad &= \quad i\Theta(x-u)\sum_{k=1}^{\infty}\operatorname{Res}\Phi_\varepsilon(\xi_k^+;x,y) \\
&\quad + i\Theta(u-x)\left(-\operatorname{Res}\Phi_\varepsilon(-i\varepsilon;x,y)-\sum_{k=1}^{\infty}\operatorname{Res}\Phi_\varepsilon(\xi_k^-;x,y)\right).
\end{aligned}
$$

Here $\Phi_\varepsilon(\xi;x,y)$ is integrand function in (2.10). Computing the residuals, we get

$$\operatorname{Res}\Phi_\varepsilon(\xi_k^{\pm};x,y)=\mp\frac{ik\sin(ky)e^{\mp(x-u)\sqrt{r^2+k^2}}}{\pi(r^2+k^2)}, \qquad \operatorname{Res}\Phi_\varepsilon(-i\varepsilon;x,y)=\frac{\sinh\left((\pi-y)r\right)}{\sinh(\pi r)}.$$

Letting $\varepsilon \to 0$ and calculating the partial derivative $\partial P^\pi/\partial x$, we can obtain the result.

2.3   The general elliptic operator

Now we extend our formula for the strip Π^l of width l and more general operator

$$
\begin{cases}
\sigma_1^2\dfrac{\partial^2 V}{\partial x^2}+2\rho\sigma_1\sigma_2\dfrac{\partial^2 V}{\partial x\partial y}+\sigma_2^2\dfrac{\partial^2 V}{\partial y^2}+\alpha_1\dfrac{\partial V}{\partial x}+\alpha_2\dfrac{\partial V}{\partial y}-r^2V=0, & (x,y)\in\operatorname{int}\Pi^l, \\
V|_{y=0}=\varphi(x), \quad x\in\mathbb{R}, \\
V|_{y=\pi}=0, \quad x\in\mathbb{R},
\end{cases}
$$

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where the coefficient ρ satisfies the condition $|\rho| < 1$. The solution of the problem (2.3) also allows the representation $V = \varphi * \hat{G}^l$ where the Green function $\hat{G}^l$ is given as follows:

$$\hat{G}^l(x,y) = \delta(s(x,y))\Theta(-y) + \frac{\pi\sigma_2^2}{(1-\rho^2)\sigma_1^2 l^2} e^{-\frac{\alpha_2 y}{2\sigma_2^2} + \beta s(x,y)} \sum_{k=0}^{\infty} k \sin\left(\frac{\pi y}{l}\right) \frac{e^{-R(k)|s(x,y)|}}{R(k)},$$

where

$$s(x,y) = \frac{\rho\sigma_1}{\sigma_2}y - x, \qquad \beta = \frac{1}{2(1-\rho^2)\sigma_1}\left(-\frac{\rho\alpha_2}{\sigma_2} + \frac{\alpha_1}{\sigma_1}\right),$$

$$R(k) = \frac{1}{2(1-\rho^2)\sigma_1}\sqrt{\frac{\alpha_1^2}{\sigma_1^2} - 2\frac{\rho\alpha_1\alpha_2}{\sigma_1\sigma_2} + \frac{\alpha_2^2}{\sigma_2^2} + 4(1-\rho^2)r^2 + \frac{4(1-\rho^2)\sigma_2^2\pi^2 k^2}{l^2}}.$$

The proof of uniqueness and existence of the solution and the construction of the Green function is a straightforward extension of arguments given in Section 4 for the Laplace operator.

3 Problem on the half-plane

3.1 Existence and uniqueness theorem

In this section we study the case of the half-plane $\Pi^\infty = \mathbb{R} \times \mathbb{R}_+$. Let us consider the elliptic boundary value problem for the operator $LV = \Delta V - r^2 V$:

$$\begin{cases} LV(x,y) = 0, & (x,y) \in \operatorname{int}\Pi^\infty, \\ V|_{y=0} = \varphi(x), & x \in \mathbb{R}, \\ V|_{y=+\infty} = 0, & x \in \mathbb{R}. \end{cases} \tag{3.1}$$

Theorem 3.1. *Let H^λ be the space of Hölder functions of order $\lambda > 0$ and let φ be a bounded function from H^λ. Then the solution of the problem (3.1) exists in the classical sense, is unique, and allows the representation*

$$V(x,y) = \int_{\mathbb{R}} \varphi(x)G^\infty(x-u,y)du. \tag{3.2}$$

where the Green function G^∞ has the form

$$G^\infty(x,y) = \delta(x)\Theta(-y) + \frac{1}{2\pi}\int_0^\infty \frac{\xi\sin(\xi y)e^{-|x|\sqrt{\xi^2+r^2}}}{\sqrt{\xi^2+r^2}}d\xi. \tag{3.3}$$

Corollary 3.1. *The Green function G^∞ can be represented in terms of the modified Bessel function $K_1(z)$ of the first order*

$$G^\infty(x,y) = \delta(x)\Theta(-y) + \frac{ry}{\pi\sqrt{x^2+y^2}}K_1\left(r\sqrt{x^2+y^2}\right). \tag{3.4}$$

Proof. Use the following identity for the Bessel functions (see [3], 3.914)

$$\int_0^\infty \frac{x\sin(ax)e^{-\beta\sqrt{\gamma^2+x^2}}}{\sqrt{\gamma^2+x^2}}dx = \frac{a\gamma}{\sqrt{a^2+\beta^2}}K_1\left(\gamma\sqrt{a^2+\beta^2}\right), \qquad \operatorname{Re}\beta, \operatorname{Re}\gamma, a > 0. \tag{3.5}$$

$\square$

3.2 Construction of the Green function

For this domain we use the similar construction of the interaction potential $P^\infty(x, y; u)$ as the limit of solutions $P_\varepsilon^\infty(x, y; u)$ to the boundary value problems

$$\begin{cases} \Delta P_\varepsilon^\infty - r^2 P_\varepsilon^\infty = 0, \\ P_\varepsilon^\infty|_{y=0} = e^{-\varepsilon(x-u)}\Theta(x-u), \\ P_\varepsilon^\infty|_{y=\infty} = 0. \end{cases} \tag{3.6}$$

Let us consider the Fourier transform

$$v_\varepsilon(y; \xi, u) = \int_\mathbb{R} e^{-i\xi x} P_\varepsilon^\infty(x, y; u)dx.$$

Then v_ε as a function of y solves the boundary value problem for the ODE

$$\begin{cases} \dfrac{\mathrm{d}^2 v_\varepsilon}{\mathrm{d}y^2} - (\xi^2 + r^2)v_\varepsilon = 0, \\ v_\varepsilon(0; \xi, u) = \dfrac{e^{i\xi u}}{\varepsilon - i\xi}, \\ v_\varepsilon(\infty; \xi, u) = 0. \end{cases} \tag{3.7}$$

It can be expressed explicitly:

$$v_\varepsilon(y; \xi, u) = \frac{e^{i\xi u - y\sqrt{\xi^2 + r^2}}}{\varepsilon - i\xi}.$$

As in the previous case we calculate the inverse Fourier transform and obtain the formula for the potential P_ε^∞:

$$P_\varepsilon^\infty(x, y; u) = \frac{1}{\pi} \int_\mathbb{R} \frac{e^{i\xi u - y\sqrt{\xi^2 + r^2}}}{\varepsilon - i\xi} d\xi$$

Integrating along the loop depicted in Figure 2 and applying the Jordan lemma we get the following representation for P_ε^∞:

$$\begin{aligned} P_\varepsilon^\infty(x - u, y) &= -\frac{\Theta(u - x)}{2\pi} \int_{L_1 + L_2} \Phi(\xi; x, y)d\xi - \frac{\Theta(x - u)}{2\pi} \int_{L_3 + L_4} \Phi(\xi; x, y)d\xi \\ &\quad - i\Theta(x - u)\mathrm{Res}\left(\Phi(-i\varepsilon)\right). \end{aligned}$$

In contrast to the strip case we have only one residue $\xi = -i\varepsilon$ and two branch points $\xi = \pm ir$. Letting $\varepsilon \to \infty$ and computing the partial derivative with respect to x, we obtain the needed formula for the Green function.

3.3 The general elliptic operator

As in the previous section we generalize the results for a more general operator

$$\begin{cases} \sigma_1^2 \dfrac{\partial^2 V}{\partial x^2} + 2\rho\sigma_1\sigma_2 \dfrac{\partial^2 V}{\partial x \partial y} + \sigma_2^2 \dfrac{\partial^2 V}{\partial y^2} \alpha_1 \dfrac{\partial V}{\partial x} + \alpha_2 \dfrac{\partial V}{\partial y} - r^2 V = 0, & (x, y) \in \mathrm{int}\,\Pi^\infty, \\ V|_{y=0} = \varphi(x), & x \in \mathbb{R}, \\ V|_{y=\pi} = 0, & x \in \mathbb{R}, \end{cases} \tag{3.8}$$

where $|\rho| < 1$. The solution has the representation $V = \varphi * \hat{G}^\infty$ where $\hat{G}^\infty$ is the Green function

$$\hat{G}^\infty(x - u, y) = \delta(s(x - u, y))\Theta(-y) + \frac{\sigma_2^2 e^{-\frac{\alpha_2 y}{2\sigma_2^2} + \beta s(x-u,y)}}{\pi(1 - \rho^2)\sigma_1^2} \int_0^\infty \xi \sin(\xi y) \frac{e^{-R_\infty(\xi)|s_0(x-u,y)|}}{R_\infty(\xi)}d\xi,$$

where

$$R_\infty(\xi) = \frac{1}{2(1 - \rho^2)\sigma_1}\sqrt{\frac{\alpha_1^2}{\sigma_1^2} - 2\frac{\rho\alpha_1\alpha_2}{\sigma_1\sigma_2} + \frac{\alpha_2^2}{\sigma_2^2} + 4(1 - \rho^2)r^2 + 4(1 - \rho^2)\sigma_2^2\xi^2}.$$
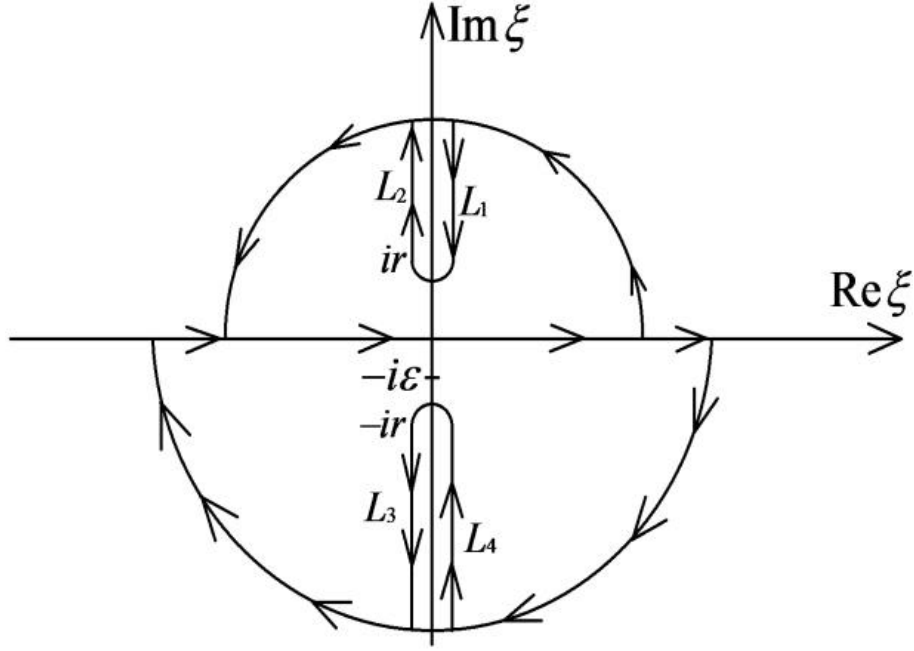
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Figure 2: The integration loop for the half-plane case

4 Proofs

4.1 Proof of Theorem 2.1

We split the arguments into four parts. We prove step by step the following propositions:

1. The convolution product (2.2) is a continuous and bounded function in domain Π^π for all bounded boundary functions φ and φ_1 from the space H^λ.

2. The convolution product (2.2) is a twice differentiable function in the interior of the domain Π^π for all bounded boundary function φ from H^λ.

3. The convolution product (2.2) is the solution of the problem (2.1)

4. The solution of the problem (2.1) is unique.

4.1.1 Proposition 1

We check the absolute convergence of integrals in the convolution (2.2), implying that the convolution is continuous function. Recall the formula from (2.2)

$$\varphi * G^\pi = \frac{1}{\pi} \int_\mathbb{R} \varphi(u) \sum_{k=0}^\infty \frac{k \sin(ky) e^{-|x-u|\sqrt{k^2+r^2}}}{\sqrt{k^2+r^2}} du + \Theta(-y) \int_\mathbb{R} \delta(x-u)\varphi(u)du.$$

Representing the first summand as the sum of integrals taken over domains $(-\infty, x]$ and $[x, -\infty)$ and using the change of variables $x - u = \xi$ in the first integral and $u - x = \xi$ in the second, we get that

$$\varphi * G = \frac{1}{\pi} \int_0^\infty (\varphi(x+\xi) - \varphi(x-\xi)) \sum_{k=0}^\infty \frac{k \sin(ky) e^{-\xi\sqrt{k^2+r^2}}}{\sqrt{k^2+r^2}} d\xi + \Theta(-y)\varphi(x).$$

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The following chain of estimates, where $\Gamma(\nu)$ is the Euler Gamma function (see [1]) and C denotes constants varying from step to step, completes the proof:

$$\int_0^\infty \left| (\varphi(x+\xi) - \varphi(x-\xi)) \sum_{k=0}^\infty \frac{k\sin(ky)e^{-\xi\sqrt{k^2+r^2}}}{\sqrt{k^2+r^2}} \right| d\xi \leq \left\{ \begin{array}{l} |\sin(ky)| \leq 1; \\ |\varphi(x+\xi) - \varphi(x-\xi)| \leq C\xi^\lambda. \end{array} \right\} \leq$$

$$\leq \int_0^\infty \left| C\xi^\lambda \sum_{k=0}^\infty \frac{ke^{-\xi\sqrt{k^2+r^2}}}{\sqrt{k^2+r^2}} \right| d\xi \leq \left\{ \begin{array}{l} \int_0^\infty x^{\nu-1}e^{-\mu x}dx = \frac{\Gamma(\nu)}{\mu^\nu}, \\ \mathrm{Re}\mu, \mathrm{Re}\nu > 0. \end{array} \right\} \leq$$

$$\leq C\Gamma(1+\lambda) \sum_{k=0}^\infty \frac{k}{(r^2+k^2)^{1+\lambda/2}} \leq C \sum_{k=0}^\infty \frac{1}{k^{1+\lambda/2}} < \infty.$$

4.1.2 Proposition 2

We consider the Dirichlet problem for the equation $LV = 0$ in the disk Ω centered at the origin with radius $\tilde{\rho}$.

$$\begin{cases} LV(x,y) = 0, & (x,y) \in \Omega, \\ V|_{\partial\Omega} = \varphi(x,y), \end{cases} \tag{4.1}$$

First we prove the following lemma:

Lemma 4.1. *The problem (4.1) has a classical solution for any continuous boundary function* $\varphi(x,y)$.

Proof. In the polar coordinates $x = \rho\cos\theta$, $y = \rho\sin\theta$, the problem (4.1) for $V(\rho,\theta)$ has the form

$$\begin{cases} \dfrac{\partial^2 V}{\partial\rho^2} + \dfrac{1}{\rho}\dfrac{\partial V}{\partial\rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 V}{\partial\theta^2} = 0, \\ V(\tilde{\rho},\theta) = \psi(\theta). \end{cases} \tag{4.2}$$

where $\psi(\theta) = \varphi(x,y)|_{(x,y)\in\partial\Omega}$. We separate the variables and solve the Sturm–Liouville problem

$$V(\rho,\theta) = \sum_{n=0}^\infty P_n(\rho)Q_n(\theta), \qquad \frac{P_n'' + P_n'/\rho - r^2 P_n}{P_n/\rho^2} = \frac{Q_n''}{Q_n} = n^2, \quad n \geq 0.$$

It is well known that the solution can be represented as follows

$$\begin{aligned} V(\rho,\theta) &= \frac{2}{\pi}\frac{I_0(r\rho)}{I_0(r\tilde{\rho})}\int_{-\pi}^\pi \psi(\xi)d\xi \\ &+ \frac{1}{\pi}\sum_{n=0}^\infty \frac{I_n(r\rho)}{I_n(r\tilde{\rho})}\left(\sin n\theta \int_{-\pi}^\pi \sin(n\xi)\psi(\xi)d\xi + \cos n\theta \int_{-\pi}^\pi \cos(n\xi)\psi(\xi)d\xi\right) \end{aligned}$$

where $I_n(z)$ is the modified Bessel function (see [1]).

It is easy to show that $I_n(z)$ has the following properties:

$$\frac{I_n(z)}{I_n(\tilde{z})} \leq \left(\frac{z}{\tilde{z}}\right)^n, \qquad \frac{I_{n+1}(z)}{I_n(\tilde{z})} \leq \left(\frac{z}{\tilde{z}}\right)^n \frac{z}{2}, \qquad \frac{I_{n-1}(z)}{I_n(\tilde{z})} \leq \left(\frac{z}{\tilde{z}}\right)^n \frac{2}{z}. \tag{4.3}$$

Indeed, using the definition of the modified Bessel function $I_n(z)$, see [1], we have:

$$\frac{I_n(z)}{I_n(\tilde{z})} = \frac{\sum_{k=0}^\infty \frac{1}{k!(n+k)!}\left(\frac{z}{2}\right)^{n+2k}}{\sum_{k=0}^\infty \frac{1}{k!(n+k)!}\left(\frac{\tilde{z}}{2}\right)^{n+2k}} \leq \left(\frac{z}{\tilde{z}}\right)^n \frac{\sum_{k=0}^\infty \frac{1}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k}}{\sum_{k=0}^\infty \frac{1}{k!(n+k)!}\left(\frac{\tilde{z}}{2}\right)^{2k}} \leq \left(\frac{z}{\tilde{z}}\right)^n,$$

$$\frac{I_{n+1}(z)}{I_n(\tilde{z})} = \frac{\sum_{k=0}^{\infty} \frac{1}{k!(n+k+1)!}\left(\frac{z}{2}\right)^{n+1+2k}}{\sum_{k=0}^{\infty} \frac{1}{k!(n+k)!}\left(\frac{\tilde{z}}{2}\right)^{n+2k}} \leq \left(\frac{z}{\tilde{z}}\right)^n \frac{z}{2} \frac{\sum_{k=0}^{\infty} \frac{1}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k}}{\sum_{k=0}^{\infty} \frac{1}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k}} \leq \left(\frac{z}{z}\right)^n \frac{z}{2},$$

$$\frac{I_{n-1}(z)}{I_n(\tilde{z})} = \frac{\sum_{k=0}^{\infty} \frac{1}{k!(n+k-1)!}\left(\frac{z}{2}\right)^{n-1+2k}}{\sum_{k=0}^{\infty} \frac{1}{k!(n+k)!}\left(\frac{\tilde{z}}{2}\right)^{n+2k}} \leq \left(\frac{z}{\tilde{z}}\right)^n \frac{2}{z} \frac{\sum_{k=0}^{\infty} \frac{1}{k!(n+k-1)!}\left(\frac{z}{2}\right)^{2k}}{\sum_{k=0}^{\infty} \frac{1}{k!(n+k-1)!}\left(\frac{z}{2}\right)^{2k}} \leq \left(\frac{z}{z}\right)^n \frac{2}{z}.$$

From here we immediately obtain the bounds for $V(\rho, \theta)$ and $\dfrac{\partial^2 V(\rho, \theta)}{\partial \theta^2}$:

$$|V(\rho, \theta)| \leq C \sum_{n=0}^{\infty} \left(\frac{\rho}{\tilde{\rho}}\right)^n \leq \text{Const}, \quad \left|\frac{\partial^2 V(\rho, \theta)}{\partial \theta^2}\right| \leq C \sum_{n=0}^{\infty} n^2 \left(\frac{\rho}{\tilde{\rho}}\right)^n \leq \text{Const}.$$

For the derivative $\partial V/\partial \rho$ we use the properties of the modified Bessel functions, see [1]:

$$I_{n-1}(r\rho) + I_{n+1}(r\rho) = 2\frac{d}{rd\rho}\left(I_n(r\rho)\right), \quad I_{n-1}(r\rho) - I_{n+1}(r\rho) = 2\frac{n}{r\rho}I_n(r\rho).$$

Thus, the derivative $\partial V/\partial \rho$ is bounded because

$$\left|\frac{\partial V}{\partial \rho}(\rho, \theta)\right| \leq C \left(\sum_{n=0}^{\infty} \frac{I_{n-1}(r\rho)}{I_n(r\tilde{\rho})} + \sum_{n=0}^{\infty} \frac{I_{n+1}(r\rho)}{I_n(r\tilde{\rho})}\right) \leq \text{Const}.$$

These relations mean that V, $\partial V/\partial \theta$, and $\partial V/\partial \rho$ are continuous functions. The arguments for the second derivative $\partial^2 V/\partial \rho^2$ are similar. $\qquad \square$

Now we can complete the proof of Proposition 2. We take an arbitrary point (x, y) from int Π^{π} and consider a disk centered in (x, y) and contained in int Π^{π}. Due to Proposition 1 the convolution (2.2) is continuous on the boundary of this disk. By Lemma 4.1 (2.2) it is twice differentiable at any point of its interior. Hence, the convolution (2.2) is a twice differentiable function in the interior of Π^{π}.

4.1.3 Proposition 3

First, we check the boundary conditions

$$V(x, y)|_{y=0} = \int_{\mathbb{R}} \varphi(u)G^{\pi}(x-u, 0)du = \varphi(x), \quad V(x, y)|_{y=\pi} = \int_{\mathbb{R}} \varphi(u)G^{\pi}(x-u, \pi)du = 0.$$

It is easy to show that the Green function is a weak solution, i.e. we understand the equality $LG^{\pi} = 0$ in the following sense:

$$LG^{\pi} = 0 \iff \left\{(G^{\pi}, Lz)_{\Omega} = \iint_{\Omega} G^{\pi} Lz \, dxdy = 0, \quad \forall z \in \dot{C}^2, \quad \forall \Omega \subsetneq \Pi^{\pi}\right\},$$

where $\dot{C}^2$ is the class of twice continuously differentiable finite functions (the class of test functions) and Ω is a compact sub-domain. Hence, for the convolution $\varphi * G^{\pi}$ we have

$$(\varphi * G^{\pi}, Lz)_{\Omega} = \iint_{\Omega} (\varphi * G^{\pi}) Lz \, dxdy = \varphi * \iint_{\Omega} G^{\pi} Lz \, dxdy = 0.$$

It is well known that the elliptic operator is self-adjoint. Therefore

$$((L(\varphi * G^{\pi}), \varphi)_{\Omega} = (\varphi * G^{\pi}, L^*z)_{\Omega} = (\varphi * G^{\pi}, L^*z)_{\Omega} = (\varphi * G^{\pi}, Lz)_{\Omega} = 0.$$

Due to Propositions 1 and 2 function the $L(\varphi * G^{\pi})$ is continuous and, therefore, bounded on the compact Ω. Hence, $L(\varphi * G^{\pi}) = 0$ because Ω is arbitrary.

4.1.4 Proposition 4

Suppose that we have two different bounded functions V and $\tilde{V}$, both solving the problem (2.1). Their difference $V - \tilde{V}$ is a bounded function solving the homogeneous problem

$$
\begin{cases}
LV_0(x, y) = 0, & (x, y) \in \text{int } \Pi^\pi, \\
V_0(x, 0) = 0, & x \in \mathbb{R}, \\
V_0(x, \pi) = 0, & x \in \mathbb{R}.
\end{cases}
\tag{4.4}
$$

Lemma 4.1. *The solution of the problem (4.4) is the infinite sum*

$$
V_0(x, y) = \sum_{k=0}^{\infty} \left(A_k e^{\sqrt{r^2 + k^2}\, x} \sin ky + B_k e^{-\sqrt{r^2 + k^2}\, x} \sin ky \right).
\tag{4.5}
$$

It is easy to show that all summands in (4.5) are unbounded functions except zero (the eigenfunction corresponding to $k = 0$). Hence, $V = \tilde{V}$.

Proof. We separate the variables $V_0(x, y) = X(x)Y(y)$ and solve the Sturm–Liouville problem:

$$
YX'' + Y''X - r^2 XY = 0, \qquad \frac{X'' - r^2 X}{X} = \frac{-Y''}{Y} = \lambda^2.
$$

Spectrum is $\lambda = \lambda_k = k$, $k \in \mathbb{N}$ and the eigenfunctions are $X_k(x)Y_k(y) = e^{\pm\sqrt{r^2 + k^2}\, x} \sin ky$. $\qquad \square$

4.2 Proof of Theorem 2.2

We put $\xi = x - u$ and consider the part of the Green function G^π given by the series:

$$
R(\xi, y, r) = \sum_{k=0}^{\infty} \frac{k \sin ky}{\sqrt{k^2 + r^2}} e^{-\xi\sqrt{k^2 + r^2}}.
\tag{4.6}
$$

In the case $r = 0$, see [3] 1.445.1, we have:

$$
R(\xi, y, 0) = \frac{\sin y}{2(\cosh \xi - \cos y)}, \qquad \frac{\partial R(\xi, y, 0)}{\partial \xi} = T(\xi, y, 0) = -\frac{\sin y \sinh \xi}{2(\cosh \xi - \cos y)^2}.
\tag{4.7}
$$

Using the formula

$$
\int_1^\infty e^{-k\xi t} J_0(\xi r \sqrt{t^2 - 1}) dt = \frac{e^{-\xi\sqrt{k^2 + r^2}}}{\xi\sqrt{k^2 + r^2}},
$$

see [3], 6.646(1)), we have

$$
R(\xi, y, r) = \xi \sum_{k=0}^{\infty} k \sin ky \int_1^\infty e^{-k\xi t} J_0\left(\xi r \sqrt{t^2 - 1}\right) dt.
$$

The change of the integration and the summation yields

$$
R(\xi, y, r) = -\xi \int_1^\infty J_0\left(\xi r \sqrt{t^2 - 1}\right) T(\xi t, y, 0) dt = -\int_1^\infty J_0\left(\xi r \sqrt{t^2 - 1}\right) d\left(R(\xi t, y, 0)\right).
$$

Using the second formula in (4.7), we finish the proof.

4.3 Proof of Theorem 3.1

The proof consists of four parts:

1. The convolution product (3.2) is a continuous and bounded function in the domain Π^∞ for all bounded boundary functions φ from H^λ.

2. The convolution product (3.2) has the continuous second-order partial derivatives in $\Pi^\infty/\partial\Pi^\infty$ for all bounded boundary functions φ from H^λ.

3. The convolution product (3.2) is the solution of the problem (3.1)

4. The solution of the problem (3.1) is unique.

We omit the proofs of Propositions 2 and 3 as the arguments are similar to those in the previous case.

4.3.1 Proposition 1

We make changes in the variables similar to those used above and get that

$$G^\infty * \varphi = \frac{1}{\pi} \iint_{\mathbb{R}^2_+} (\varphi(x+\eta) - \varphi(x-\eta)) \frac{\xi \sin(\xi y) e^{-\eta\sqrt{r^2+\xi^2}}}{\sqrt{r^2+\xi^2}} d\eta d\xi + \Theta(-y)\varphi(x).$$

Using the bounds relations we infer the absolute convergence of the integral:

$$\iint_{\mathbb{R}^2_+} |\varphi(x+\eta) - \varphi(x-\eta)| \frac{\xi |\sin(\xi y)| e^{-\eta\sqrt{r^2+\xi^2}}}{\sqrt{r^2+\xi^2}} d\eta d\xi \le$$

$$\le \left\{ \begin{array}{l} |\sin(ky)| \le 1; \\ |\varphi(x+\xi) - \varphi(x-\xi)| \le C\xi^\lambda \end{array} \right\} \le C \iint_{\mathbb{R}^2_+} \eta^\lambda \frac{\xi e^{\eta\sqrt{r^2+\xi^2}}}{\sqrt{r^2+\xi^2}} d\eta d\xi \le$$

$$\le \left\{ \begin{array}{l} \int_0^\infty x^{\nu-1} e^{-\mu x} dx = \frac{\Gamma(\nu)}{\mu^\nu}, \\ \operatorname{Re}\mu, \operatorname{Re}\nu > 0 \end{array} \right\} \le C\Gamma(1+\lambda) \int_0^\infty \frac{\xi d\xi}{(r^2+\xi^2)^{1+\lambda/2}} \le C \int_0^\infty \frac{d\xi}{\xi^{1+\lambda/2}} \le C.$$

4.3.2 Proposition 4

As in the previous case we suppose that there are two different bounded solutions V and $\tilde{V}$ of the problem (3.1). Then we have the homogeneous boundary problem for the difference $V_0 = V - \tilde{V}$.

$$\left\{ \begin{array}{ll} \Delta V_0 - r^2 V_0 = 0, & (x,y) \in \operatorname{int} \Pi^\infty, \\ V_0(x,0) = 0, & x \in \mathbb{R}, \\ V_0(x,\infty) = 0, & x \in \mathbb{R}. \end{array} \right. \tag{4.8}$$

We show that the solution of (4.8) can be represented in terms of the modified Bessel functions. After that we use the asymptotic of modified Bessel functions and show that only the zero function solves the problem (4.8) in the class of bounded functions. The proof of the lemma below completes the proof of Proposition 4.

Lemma 4.2. *The solution of the problem (4.8) is the infinite sum*

$$V_0(x,y) = \sum_{n=0}^\infty (A_n I_n(r\nu) \sin ny + B_n K_n(r\nu) \sin ny). \tag{4.9}$$

This sum does not have bounded summands for $n \ge 1$.

Proof. In the polar coordinates $x = \nu\cos\varphi$ and $y = \nu\sin\varphi$ we have the following Sturm–Liouville problem in separated variables $V = N(\nu)\Phi(\varphi)$:

$$\frac{N'' + N'/\nu - r^2 N}{N/\nu^2} = -\frac{\Phi''}{\Phi} = \lambda^2.$$

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For the phase component Φ we have the spectral problem

$$\begin{cases} \Phi'' + \lambda^2 \Phi = 0, \\ \Phi(0) = 0, \\ \Phi(\pi) = 0. \end{cases} \qquad (4.10)$$

The eigenfunctions are $\Phi_k(\varphi) = \sin k\varphi$ and $\lambda = \lambda_k = k$ with $k \in \mathbb{N}$. For the radial component we have the modified Bessel equation

$$N_k'' + \frac{N_k'}{\nu} - r^2 N_k - \frac{k^2}{\nu^2} N_k = 0.$$

Therefore, the solution of the problem (4.8) can be represented in the form from (4.9). The modified Bessel functions have the well-known asymptotic, see [1],

$$K_n(z) \sim \infty, \quad I_n(z) \sim 0, \quad (z \to 0), \qquad K_n(z) \sim 0, \quad I_n(z) \sim \infty, \quad (z \to \infty).$$

Hence, in the sum we do not have bounded summands for $n \geq 1$. $\qquad \square$

References

[1] Abramowitz M., Stegun I., Eds. (1972) *Handbook of mathematical functions with formulas, graphs and mathematical tables.* Dover, New York.

[2] Broadie M., Detemple J. (1997) *The valuation of american options on multiple assets* Mathematical Finance, Vol. 7 No.3 241-286.

[3] Gradshteyn I.S., Ryzhik I.M. (1980) *Tables of integrals, series and products. Academic Press, New York.*

[4] Danilov V.G., Omel'yanov G.A. (1999) *Estimate for the width mushy region between two liquids of different densities* Preprint. Portugal: Universidade da Beira Interior.

[5] Miranda C. (1970) *Partial differential equations of elliptic type.* Springer-Verlag.

[6] Melnikov Yu.A. (2011) *Construction of Green's functions for the Two-Dimensional Static Klein-Gordon Equation.* J. Part. Diff. Eq. Vol. 24, No. 2, pp. 114-139.

[7] Vladimirov V.S. (1971) *Equations of mathematical physics.* Marcel Dekker.